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AN APPROACH FOR CONTINUOUS METHOD FOR THE GENERAL CONVEX PROBLEM

PRASHANT CHAUHAN

Department of Mathematics, S. L. Education Institute, Moradabad, Uttar Pradesh, India

ABSTRACT

In this paper, we introduce a new continuous method for the general convex problem (1). Our method overcomes all the shortfalls of the previous methods, yet maintains strong convergence. As pointed out in [2] that the projection model (3) is difficult or impossible to be implemented for a general convex constraint, our approach first transforms problem (1) into a monotone variational inequality problem. Then, a continuous method is introduced for this variational inequality problem. Our continuous model differs from the one in [3] in two aspects. First, we employ a merit function in our continuous model. The structure of our model is similar to the one in [23, 9]. Second, the bound for the scaling parameter (called α_0 in [3]) is constant in [3], but it is variable in our continuous model; in particular, an ODE for this variable as function of time is established.

KEYWORDS: Convex Programming, Variational Inequality, Continuous Method

1. INTRODUCTION

In this paper, we consider the following convex programming problem:

(CP) $\min_{x\in R^n} f(x)$ (1a)

Such that $g(x)\le 0$, $g : R^n \to R^m$ (1b)

Where f(x) and g(x) are convex functions and have continuous first-order derivatives. Obviously, for convex problem (1), every local solution is also a global solution. For problem (1), we define

$$\Omega_0 = \left\{ x\in R^n \,\middle|\, g(x)\le 0 \right\}$$

as the feasible set and

$$\Omega_0^* = \left\{ x\in R^n \,\middle|\, x \text{ is an optimal solution of } (1) \right\}$$

as the solution set. Throughout this chapter, we assume that there exists a

finite $x^*\in \Omega_0^*$ and that the Slater condition [i.e., $\exists$ an x such that $g(x)<0$, [5]] is satisfied. Thus, the KKT conditions for problem (1) can be written as

$$\nabla f(x) + (g'(x))^T = 0, \tag{2a}$$

$$y\ge 0\ ,\ g(x)\le 0\ ,\ y^T g(x) = 0 \tag{2b}$$

Even though continuous methods can be traced back to the 1950s [1], the most significant theoretical contribution to continuous methods comes from Hopfield [16–18]. In recent years, the restriction on the merit function being a Lyapunov function has been removed to allow wider applications [25, 20, 10].

As in Flam [7] the ODE

$dx(t)/dt = -\lambda x(t)$, $\lambda > 0$.

The solution of the above ODE is

$x(t) = x_0 e^{-\lambda t}$

Thus, it is easy to see that $x(t) \to 0$ as $t \to \infty$. But x(t) will never reach $x^* = 0$ in any finite time if $x_0 \neq 0$. Therefore, the conditions and result of theorem 3.2 in [7] do not hold. The first convergence proof of a continuous model for the general convex problem (1) was given in [2]. The ODE suggested in [2] is based on the gradient projection method and has the following form:

$$dx(t)/dt = -[x - P_{\Omega_0}(x - \nabla f(x))] \tag{3}$$

Where Ω_0 the feasible is set of (1) and $P_{\Omega_0}(\cdot)$ is the projection onto Ω_0. Even though this method is theoretically sound, it has serious practical difficulties, since the projection $P_{\Omega_0}(\cdot)$ is difficult to compute a general convex set Ω_0.

2. PROBLEM EQUIVALENT TO VI

First, let us look at the following lemma,

Lemma 2.1. $x \in \Omega_0 \Leftrightarrow \exists a y \in R^m$ such that x and y satisfy (2).

Proof. By [22] for $\Rightarrow$ and Theorem in [8] for $\Leftarrow$.

Now, let us define

$$u = \begin{pmatrix} x \\ y \end{pmatrix},\ F(u) = \begin{pmatrix} \nabla f(x) + (g'(x))^T y \\ -g(x) \end{pmatrix}, \tag{4}$$

and consider the following VI problem:

$$(VI(\Omega, F)) \text{ Find } u^* \in \Omega \text{ such that } (u - u^*)^T F(u^*) \geq 0,\ \forall u \in \Omega \tag{5}$$

Where

$$\Omega = \left\{ u = \begin{pmatrix} x \\ y \end{pmatrix} \middle| x \in R^n, y \in R_+^m \right\}, R_+^m = \left\{ y \in R_+^m \middle| y \geq 0 \right\}.$$

Then, the following lemma shows that F (u) defined in (4) is monotone.

Lemma 2.2. F (u) in (4) is monotone on Ω.

Proof. For any $u=(x^T,y^T)$, $\bar{u}=(\bar{x}^T,\bar{y}^T)^T\in\Omega$ we have

$$(u-\bar{u})^T(F(u)-F(\bar{u})) = \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}^T \left(\nabla f(x)-\nabla f(\bar{x})+(g'(x))^T y-(g'(\bar{x}))^T \bar{y}\right)$$

$$=(x-\bar{x})^T(\nabla f(x)-\nabla f(\bar{x}))+y^T\{g(\bar{x})-g(x)-g'(x)(\bar{x}-x)\} \quad +\bar{y}^T\{g(x)-g(\bar{x})-g'(\bar{x})(x-\bar{x})\}$$

(6) Since f(x) is convex, we have

$$(x-\bar{x})^T(\nabla f(x)-\nabla f(\bar{x}))\geq 0 \;\; (x-\bar{x})\geq 0 \tag{7}$$

Since g(x) is convex, from Proposition 4 in [21], we have

$$g(x)\geq\{g(\bar{x})+g'(x)(\bar{x}-x)\}, \tag{8a}$$

$$g(\bar{x})\geq\{g(x)+g'(x)(x-\bar{x})\} \tag{8b}$$

Replacing (7) and (8) into (6) and noticing that $y\geq 0$ and $\bar{y}\geq 0$, we have

$$(u-\bar{u})^T(F(u)-F(\bar{u}))\geq 0.$$

This completes the proof of Lemma 2.2.

Lemma 2.2 indicates that (5) is a monotone VI problem. Let $\Omega*$ be the optimal solution set of (5).

Then, we have the following lemma.

Lemma 2.3. u satisfies (2) if and only if $u\in\Omega*$.

Proof. Let $u=(x^T,y^T)^T$

(⇒) Since u satisfies (2), then $u\in\Omega$ and, $\forall\bar{u}=(\bar{x}^T,\bar{y}^T)^T\in\Omega$,we have $(\bar{u}-u)^T F(u)=-(\bar{y}-y)^T g(x)$

$$=-\bar{y}^T g(x) \tag{9}$$

Obviously, $\bar{y}\geq 0$ .Then, $g(x)\leq 0$ and (9) imply $(\bar{u}-u)^T F(u)\geq 0$, $\forall\bar{u}\in\Omega$.

(⇐) Since $u\in\Omega*$, wehave $y\geq 0$ and

$$(\bar{x}-x)^T[\nabla f(x)+(g'(x))^T y] -(\bar{y}-y)^T g(x)\geq 0 \;\forall\bar{u}\in\Omega \tag{10}$$

Let the vectors $\bar{y}=y$ and $\bar{x}$ in R^n. Then, (10) implies that

$$\nabla f(x)+(g'(x))^T y=0$$

Consequently,

$$-(\bar{y}-y)^T g(x) \geq 0\,,\ \forall \bar{y} \geq 0 \tag{11}$$

In (11), let

$\bar{y} = y + e_i\,,\ \ i = 1,2,......m$ where e_i is the ith column vector of the identity matrix I_m. Then, (11) becomes $-e_i^T g(x) \geq 0\,, i = 1,2,......m$.

These inequalities imply that g(x) ≤ 0. Now, we need only to verify that $y^T g(x) = 0$. Since y ≥ 0 and (11) is true for all $\bar{y} \geq 0$, setting $\bar{y} = 0, 2y$ respectively, we obtain

$y^T g(x) \geq 0$ and $-y^T g(x) \geq 0$.

This implies that $y^T g(x) = 0$. Therefore, u satisfies (2). This proves Lemma 2.3.

3. PROJECTION

For any $v \in R^{m+n}$, we denote by $P_\Omega(v)$ the projection of v on Ω under the Euclidean norm. In other words, $P_\Omega(v) = \arg\min\{\|v-u\| | u \in \Omega\}$ is the unique solution of the minimization problem $\min\{\|v-u\| | u \in \Omega\}$. Notice that the projection on Ω [5] formulated from the convex programming problem is trivial.

A basic property of the projection mapping on a closed convex set is that

$$(v - P_\Omega(v))^T (u - P_\Omega(v)) \leq 0\ ,\ \forall v \in R^n\,,\ \forall u \in \Omega \tag{12}$$

Since the early work of Eaves [6] and others, we know that a variational inequality VI (Ω, F) is equivalent to the following projection equation:

$$u = P_\Omega[u - F(u)] \tag{13}$$

In other words, solving VI (Ω, F) is equivalent to solving the nonsmooth equation (13), i.e., finding the zero point of the residue function

$$e(u) = u - P_\Omega[u - F(u)] \tag{14}$$

Because $e(u)$ is continuous and since

$$e(u) = 0 \Leftrightarrow \forall u \in \Omega^*$$

Let $u^* \in \Omega^*$ be a solution. For any $u \in R^n$, $P_\Omega[u - F(u)] \in \Omega$. It follows from (5) that

$$F(u^*)^T \{P_\Omega[u - F(u)] - u^*\} \geq 0\ ,\ \forall u \in R^n \tag{15}$$

Setting $v = u - F(u)$ and $u = u^*$ in (12), and using the notation $e(u)$, we obtain

$$\{e(u) - F(u)\}^T \{P_\Omega[u - F(u)] - u^*\} \geq 0\ ,\ \forall u \in R^n \tag{16}$$

Provided that the mapping F is monotone with respect to Ω , we have

$$\{F(P_\Omega[u-F(u)])-F(u^*)\}^T\{P_\Omega[u-F(u)]-u^*\}\geq 0,\ \forall u\in R^n \tag{17}$$

(15) Follows from the definition of variational inequalities; (16) follows from a basic property of the projection mapping; (17) follows from the assumption of monotonicity of the mapping F.

Besides $e(u)$ [7],

$$d(u)=e(u)-\{F(u)-F(P_\Omega[u-F(u)])\} \tag{18}$$

Adding (15), (16), (17) and using the notation $e(u)$ and $d(u)$ we have

$$d(u)^T((u-u^*)-e(u))\geq 0,\forall u\in R^n$$

and it follows that

$$(u-u^*)^T d(u)\geq e(u)^T d(u),\forall u\in R^n \tag{19}$$

Lemma 3.1. For $L\in(0,1)$, if the following condition holds:

$$\|F(u)-F(P_\Omega[u-F(u)])\|\leq L\|e(u)\|,\forall u\in R^{n+m} \tag{20}$$

then we have

$$e(u)^T d(u)\geq(1-L)\|e(u)\|^2 \tag{21}$$

Proof. From (18) and (20), it is easy to see that

$$e(u)^T d(u)=\|e(u)\|^2-e(u)^T\{F(u)-F(P_\Omega[u-F(u)])\}$$

$$\geq\|e(u)\|^2-\|e(u)\|\|F(u)-F(P_\Omega[u-F(u)])\|$$

$$\geq(1-L)\|e(u)\|^2$$

This proves (21) and Lemma 3.1.

For some positive scalar β we consider the mapping βF and denote

$$e(u,\beta)=u-P_\Omega[u-\beta F(u)]$$

$$d(u,\beta)=e(u,\beta)-(\beta F(u)-\beta F(P_\Omega[u-\beta F(u)])).$$

It is trivial to see that, for any $\beta>0$,

$$e(u,\beta)=0\Leftrightarrow e(u)=0\Leftrightarrow u\in\Omega^*.$$

Now, we explore some properties for $e(u,\beta)$.

Lemma 3.2. For all $u \in R^n$ and $\bar{\beta} \geq \beta > 0$, it holds that

$$\|e(u,\bar{\beta})\| \geq \|e(u,\beta)\| \tag{22}$$

$$\|e(u,\bar{\beta})\| / \bar{\beta} \leq \|e(u,\beta)\| / \beta \tag{23}$$

Proof. If $e(u,\beta)=0$, then $e(u,\bar{\beta})=0$. Therefore the results hold. Now, we assume $e(u,\beta)\neq 0$.

Let $t = \dfrac{\|e(u,\bar{\beta})\|}{\|e(u,\beta)\|}$

We need to prove only that $1 \leq t \leq \dfrac{\bar{\beta}}{\beta}$. Note that its equivalent expression is

$$(t-1)(t-\frac{\bar{\beta}}{\beta}) \leq 0 \tag{24}$$

From (12), we have

$$(v-P_\Omega(v))^T (P_\Omega(v)-w) \geq 0,\ \forall w \in \Omega \tag{25}$$

Substituting

$$w = P_\Omega[u-\beta F(u)],\, v = u-\beta F(u)$$

in (25) and using

$$P_\Omega[u-\beta F(u)] - P_\Omega[u-\bar{\beta}F(u)] = e(u,\bar{\beta}) - e(u,\beta),$$

We get

$$\{e(u,\beta)-\beta F(u)\}^T \{e(u,\bar{\beta})-e(u,\beta)\} \geq 0 \tag{26}$$

Similarly, we have

$$\{\bar{\beta}F(u)-e(u,\bar{\beta})\}^T \{e(u,\bar{\beta})-e(u,\beta)\} \geq 0 \tag{27}$$

Multiplying (26) by $\bar{\beta}$, (27) by β, and adding the resulting expressions, we get

$$\{\bar{\beta}e(u,\beta)-\beta e(u,\bar{\beta})\}^T \{e(u,\bar{\beta})-e(u,\beta)\} \geq 0 \tag{28}$$

Consequently,

$$\bar{\beta}\|e(u,\beta)\|^2 + \beta\|e(u,\bar{\beta})\|^2 \leq (\beta+\bar{\beta})e(u,\beta)^T e(u,\bar{\beta}) \tag{29}$$

Dividing (29) by $\|e(u,\beta)\|^2$, we obtain

$$\bar{\beta}+\beta t^2 \le (\beta+\bar{\beta})t$$

thus, (24) holds and Lemma 3.2 is proved.

Lemma 3.3. For any $u \in R^{n+m}$ with $e(u) \neq 0$ and $L \in (0,1)$, there always exists a small $\bar{\beta} > 0$ such that

$$\|\beta F(u)] - \beta F(P_\Omega[u-\beta F(u)])\| \le L\|e(u,\beta)\|, \ \forall \beta \in (0,\bar{\beta}) \quad (30)$$

Proof. We divide the proof into two cases.

Case 1: $u \notin \Omega$. Because Ω is a closed convex set, there exist an $\varepsilon > 0$ and a $\delta > 0$ such that

$$\|e(u,\beta)\| > \varepsilon, \ \forall \beta \in (0,\delta) \quad (31)$$

Then, from the continuity of F , (30) can be ensured.

Case 2: $u \in \Omega$ Since $e(u) \neq 0$, there must exist a $\bar{\beta} > 0$ such that $\|e(u,\bar{\beta})\| > 0$. Therefore,

$$\|e(u,\bar{\beta})\| / \bar{\beta} > 0.$$

However $u \in \Omega$ implies

$$\lim_{\beta \to +0} \|F(u)] - F(P_\Omega[u-\beta F(u)])\| = 0$$

Together with

$$\|e(u,\bar{\beta})\| / \bar{\beta} > 0$$

and (23), this indicates that (30) is true. Combining the two cases, there exists a $\bar{\beta} > 0$ such that (30) holds. This proves Lemma 3.3.

The result of Lemma 3.3 ensures that, by reducing β small enough, (30) can be always satisfied. Then, similarly to (19) and (21), we have

$$(u-u^*)^T d(u,\beta) \ge e(u,\beta)^T d(u,\beta) \ge (1-L)\|e(u,\beta)\|^2 \quad (32)$$

4. MONOTONE VI PROBLEM

In the continuous method, we adopt a technique to adjust the parameter β so that (30) and furthermore (32) can be satisfied. We define

$$\beta = e^{\alpha} \quad (33)$$

and let

$$w=\begin{pmatrix}u\\ \alpha\end{pmatrix}, \frac{dw}{dt}=\begin{pmatrix}\frac{du}{dt}\\ \frac{d\alpha}{dt}\end{pmatrix}.$$

Let

$$T_1(u,F,\alpha)=\left\|u-P_\Omega[u-e^\alpha F(u)]\right\|^2,$$

$$T_2(u,F,\alpha)=\left\|e^\alpha F(u)-e^\alpha F(P_\Omega[u-e^\alpha F(u)])\right\|^2$$

Then, we define

$$\frac{du}{dt}=\begin{cases}-d(u,e^\alpha), & \text{if } T_2\le 0.9T_1\\ 0, & otherwise\end{cases}$$

$$\frac{d\alpha}{dt}=-\eta(u,F,\alpha)$$

Where

$$\eta(u,F,\alpha)=\begin{cases}0, \text{if } T_2\le 0.9T_1\\ 1, otherwise\end{cases}$$

Similarly, we can define

$$\xi(u,F,\alpha)=\begin{cases}1, \text{if } T_2\le 0.9T_1\\ 0, otherwise\end{cases}$$

Therefore, we have

$$\frac{dw}{dt}=-\begin{pmatrix}\xi(u,F,\alpha)d(u,e^\alpha)\\ \eta(u,F,\alpha)\end{pmatrix} \tag{34}$$

In (34), the right-hand-side term is not continuous. To make this term continuous, we can define

$$\frac{dw}{dt}=-\begin{pmatrix}\overline{\xi}(u,F,\alpha)d(u,e^\alpha)\\ \overline{\eta}(u,F,\alpha)\end{pmatrix} \tag{35}$$

Where

$$\overline{\xi}(u,F,\alpha)=\begin{cases}1, & \text{if } T_0\le 0.8\\ (0.9-T_0)10, & \text{if } 0.8\le T_0\le 0.9\\ 0, & \text{if } T_0\ge 0.9\end{cases} \tag{36a}$$

$$\bar{\eta}(u,F,\alpha)=\begin{cases}0, & \text{if } T_0\le 0.8\\ (T_0-0.8)10, & \text{if } 0.8\le T_0\le 0.9\\ 1, & \text{if } T_0\ge 0.9\end{cases} \tag{36b}$$

$$T_0(u,F,\alpha)=\frac{T_2(u,F,\alpha)}{T_1(u,F,\alpha)} \tag{36c}$$

Obviously, the right-hand-side term in (35) is continuous now.

The merit function is given as

$$E\left(w\right)=(1/2)\left\|u-u*\right\|^2,\ \forall u\in R^{n+m} \text{ where } u*\in \Omega* \text{ and is finite.} \tag{37}$$

5. CONVERGENCE PROPERTIES

Here we will discuss the convergence properties for (35), we use T_0 in place of $T_0(u,F,\alpha)$.

Theorem 5.1. For any $w_0\in R^{n+m+1}$, there exists a solution $w(t)$ of (35), with $w(t=0)=w_0$ and $w(t)$ defined in $[0,\infty)$.

Proof. From our assumption that there exists a finite $x*\in \Omega_0^*$, we

Know that there exists a finite $u*\in \Omega*$. Then,

$$\frac{d\left\|u-u*\right\|^2}{dt}=\begin{cases}-(u-u*)^T d(u,e^{\alpha}), & \text{if } T_0\le 0.8\\ -(9-10T_0)(u-u*)^T d(u,e^{\alpha}), & \text{if } T_0\le 0.8\\ 0, & \text{if } T_0>0.9\end{cases} \tag{38}$$

For all cases of (38), from (32) with L =0.9, it is easy to see

$$\frac{d\left\|u-u*\right\|^2}{dt}\le 0 \tag{39}$$

(39) Indicates that

$$u(t)\in B(u_0,u*)=\left\{u\in R^{n+m}\,\middle|\,\left\|u-u*\right\|\le\left\|u_0-u*\right\|\right\}$$

The set $B(u_0,u*)$ is a closed bounded set. In addition, it is easy to see that $\frac{d\alpha}{dt}\le 0$. From the continuity of $d(u,e^{\alpha})$, it is easy to verify that the right-hand-side function of (35) is bounded. From the Cauchy-Peano theorem, the result is obtained. This proves Theorem 5.1.

Lemma 5.1 For any $w_0\in R^{n+m+1}$, let $w(t)$ be a solution of (35) with $w(t=0)=w_0$. Then there exists a

$\bar{t} > 0$ such that

$$T_0 \leq 0.8 \text{ if } t \geq \bar{t} \tag{40}$$

Proof. For some $t \geq 0$, if $T_0 = 0$, this ensures that $u(t) \in \Omega^*$. Therefore, (40) is true. Now, we assume that $u(t) \notin \Omega^*$, $\forall t \geq 0$.

First, from (35), it is easy to see that $\frac{d\alpha}{dt} \leq 0$. Therefore, α is always monotonically non-increasing in t. From Lemma 3.3 and the definitions of $\bar{\eta}(u, F, \alpha)$ and β, we have that, $\forall u(t) \notin \Omega^*$, $\exists\ \bar{\alpha}(u)$ such that

$$T_0 \leq 0.8, \forall \alpha \leq \bar{\alpha}(u) \tag{41}$$

From the proof of Theorem 5.1, we know that $u(t) \in B(u_0, u^*)$, which is a closed bounded set. This proves Lemma 5.1.

Theorem 5.2. For any $w_0 \in R^{n+m+1}$, let $w(t)$ be a solution of (35) with $w(t=0) = w_0$. Then, $\lim_{t\to\infty} u(t)$ exists and $\lim_{t\to\infty} u(t) \in \Omega^*$.

Proof. From Lemma 5.1, there exists a $\bar{t} \geq 0$ such that $T_0 \leq 0.8, \forall t \geq \bar{t}$. From (35), (38), and (32), we can see easily that

$$\frac{d\|u - u^*\|^2}{dt} \leq (1 - L)\left\|e(u, e^{\alpha})\right\|^2, \forall t \geq \bar{t} \tag{42}$$

Where $u^* \in \Omega^*$ is finite and L is a constant in (0, 1).

From the discussion in the proof of Lemma 5.1, we know that α is monotonically non-increasing and that

$$\frac{d\alpha}{dt} = 0, \forall t \geq \bar{t}.$$

Therefore, $\lim_{t\to\infty} \alpha$ exists. Let α^* be the limit. Then, we define

$$w^* = \begin{pmatrix} u^* \\ \alpha^* \end{pmatrix}$$

From (42), we have

$$\frac{d\|w - w^*\|^2}{dt} \leq -(1 - L)\left\|e(u, e^{\alpha})\right\|^2, \forall t \geq \bar{t} \tag{43}$$

From [24] and (43) we know that

$$\lim_{t\to\infty} e(u(t),\ e^{\alpha^*}) = 0 \text{ or } \lim_{t\to\infty} e(u(t)) = 0$$

From the proof of Theorem 5.1, we know that $u(t) \in B(u_0, u^*)$, which is a compact set. Therefore, there exists a sequence t_k, with $t_k \to \infty$ as $k \to \infty$, such that $\lim_{k\to\infty} u(t_k)$ exists. Let $\bar{u}$ be the limit. From (42), it is easy to see that

$e(\bar{u}, e^{\alpha^*}) = 0$. But $e(u,\beta) = 0 \Leftrightarrow e(u) = 0 \Leftrightarrow u \in \Omega^*$. Therefore, $\bar{u} \in \Omega^*$

By replacing u^* by $\bar{u}$ in (42), we have

$$\frac{d\|u-\bar{u}\|^2}{dt} \le -(1-L)\|e(u,e^{\alpha})\|^2, \ \forall t \ge \bar{t}$$

This and $\lim_{k\to\infty} u(t_k) = \bar{u}$ imply $\lim_{t\to\infty} u(t) = \bar{u}$

This completes the proof of Theorem 5.2.

7. CONCLUSIONS

In this paper, we have discussed a continuous method for convex programming (CP) problems, which includes both a merit function and an ordinary differential equation (ODE), for the resulting variational inequality problem. Our approach is different from existing continuous methods in the literature. Strong convergence results of our continuous method are obtained under very mild assumptions. No Lipschitz condition is required in our convergence results.

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